Speed, Accuracy and Caution: the Timing of Choices Under Risk Aversion

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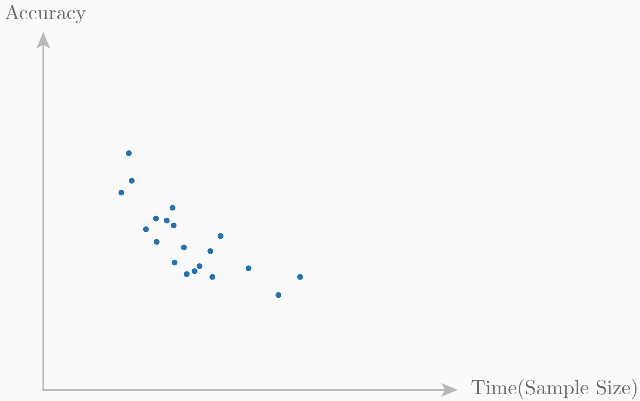
Motivation

- Large body of literature finds a negative correlation between speed and accuracy, Fried and Peterson (1969) , Swensson (1972), Luce et al. (1986), Ratcliff and Smith (2004), Ratcliff and McKoon (2008), Brown et al. (2011), Reshidi et al. (2022).



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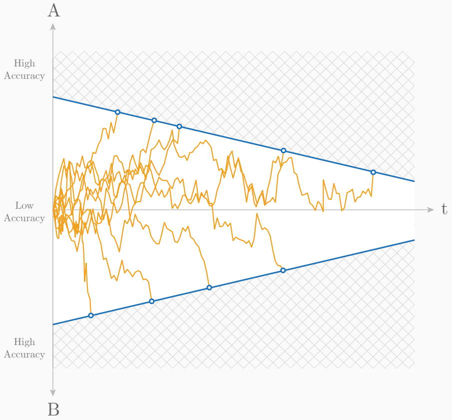

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- Which can be explained by decreasing optimal-stopping thresholds in time. This has led to multiple alternative-specifications or behavioral explanations. To name a few:
  - Bias against “throwing good money after bad” Fried and Peterson (1969).
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  - Non-stationary time discounting; Subjective costs Brown et al. (2011).
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- We ask: **Could risk-aversion alone lead to time dependent stopping rules?**

Goals of this Paper

- Analyze the richness of optimal boundaries under risk averse utility functions.
- Provide a method which is informative of the optimal boundaries.

Static Problem

Static Setup

There is an underlying state $\omega \in \{A, B\}$. Unobserved by the agent; agent has a prior $P(\omega = A) = p_0$.
Agent chooses among two alternatives a or b.
Receives a bonus x if choice matches state, no bonus otherwise.
Agent has initial wealth w.

Agent chooses how much information to acquire.
Information:

$$S_t = \begin{cases} \tilde{\mu}t + \rho W_t & \textit{if} \quad \omega = A \\ -\tilde{\mu}t + \rho W_t & \textit{if} \quad \omega = B \end{cases}$$

Timing: Choose $t \rightarrow$ given information choose a or b.
Agent pays cost $c \cdot t$ for information collection.

$$u(t, a|\omega = A) = u(w + x - ct) \qquad u(t, a|\omega = B) = u(w - ct)$$

$u(\cdot)$ - some concave utility function.

Define $\mu = \frac{2\tilde{\mu}^2}{\rho^2}$. Set $p_0 = 1/2$ for ease of exposition.

$$\max_t \mathbb{E}\left[\frac{1}{2}\left(\mathbb{P}\left(\frac{e^{S_t(\mu)}}{1+e^{S_t(\mu)}} \geq \frac{1}{1+e^{S_t(\mu)}}\right) u(w+x-ct) + \mathbb{P}\left(\frac{e^{S_t(\mu)}}{1+e^{S_t(\mu)}} < \frac{1}{1+e^{S_t(\mu)}}\right) u(w-ct)\right)\right.$$
$$\left. + \frac{1}{2}\left(\mathbb{P}\left(\frac{e^{S_t(-\mu)}}{1+e^{S_t(-\mu)}} < \frac{1}{1+e^{S_t(-\mu)}}\right) u(w+x-ct) + \mathbb{P}\left(\frac{e^{S_t(-\mu)}}{1+e^{S_t(-\mu)}} \geq \frac{1}{1+e^{S_t(-\mu)}}\right) u(w-ct)\right)\right]$$

Static Information Acquisition Problem

$$\max_{t} \; p(t)\; u(w + x - ct) + (1 - p(t))\; u(w - ct)$$

$$p(t) = \frac{1}{2}\left(\text{erf}\left(\frac{\sqrt{\mu t}}{2}\right) + 1\right)$$

f.o.c

$$\frac{u(w + x - ct) - u(w - ct)}{p(t)u'(w + x - ct) + (1 - p(t))u'(w - ct)} p'(t) = c$$

Wealth Independent Boundary

Proposition

t^* is independent of $w \iff u$ is CARA.

$$u(\cdot) = \frac{1 - e^{-\alpha(\cdot)}}{\alpha}$$

$$\frac{\int_{w-ct}^{w+x-ct} u'(s)ds}{p(t)u'(w+x-ct) + (1-p(t))u'(w-ct)} p'(t) = \frac{e^{2\alpha x} - 1}{2\alpha \left(p(t) - (p(t)-1)e^{2\alpha x}\right)} p'(t) = c$$

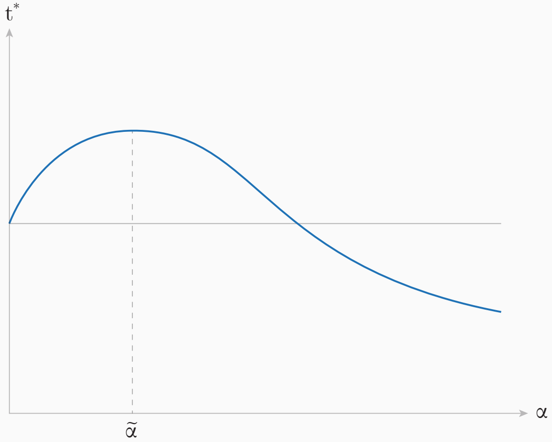
$t^*(\alpha)$ is implicitly defined by

$$\frac{\mu e^{-\frac{\mu t^*}{4}}}{4\sqrt{\pi \mu t^*}\left(\frac{1}{2}\alpha \text{erfc}\left(\frac{\sqrt{\mu t^*}}{2}\right) + \frac{\alpha}{e^{\alpha x} - 1}\right)} = c$$

t^* is a function of only α, x, c and μ. From the implicit function theorem $\exists\, \tilde{\alpha} > 0$ such that

$$\frac{\partial t^*}{\partial \alpha} > 0 \quad if\ \alpha < \tilde{\alpha} \qquad\qquad \frac{\partial t^*}{\partial \alpha} \leq 0 \quad if\ \alpha \geq \tilde{\alpha}$$

CARA Optimal Boundary



$$u(\cdot) = \sum_i \gamma_i CARA_{\alpha_i}(\cdot) \qquad \gamma_i \geq 0 \qquad \sum_i \gamma_i = 1$$

Without loss assume $\alpha_i \leq \alpha_{i+1}$. Let t_i^* be the arg max under $CARA_{\alpha_i}(\cdot)$.

Proposition

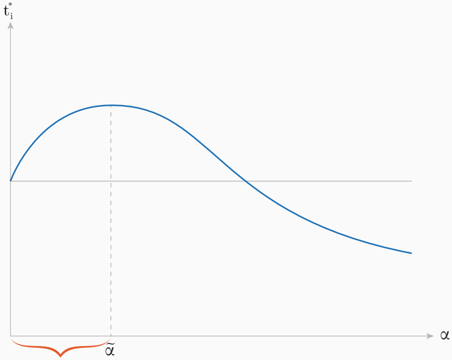
$$\forall w \;\; \min_i t_i^* \leq t^*(w) \leq \max_i t_i^*$$

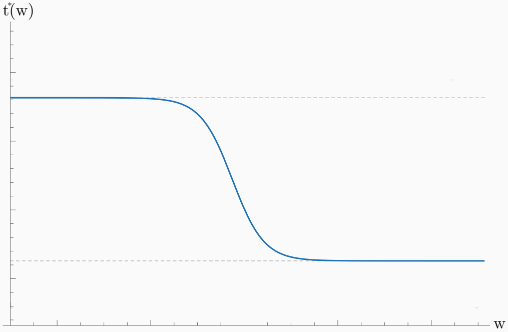
$$\lim_{w \to \infty} t^*(w) = t_1^* \qquad \lim_{w \to -\infty} t^*(w) = t_I^*$$

$$t^*(w) \begin{cases} \text{may be quasiconcave} & \text{if } \min_i \alpha_i \leq \tilde{\alpha} \leq \max_i \alpha_i \\ \text{monotonically converges} & \text{otherwise} \end{cases}$$

Static mixed-CARA

$$u(\cdot) = \sum_i \gamma_i CARA_{\alpha_i}(\cdot) \qquad \gamma_i \geq 0 \qquad \sum_i \gamma_i = 1$$





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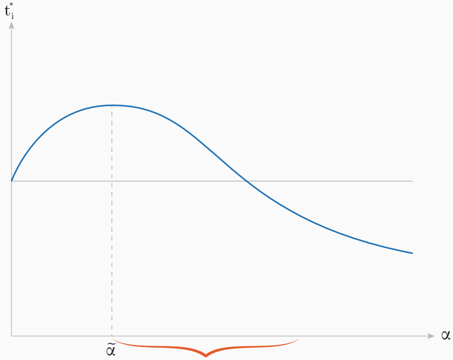

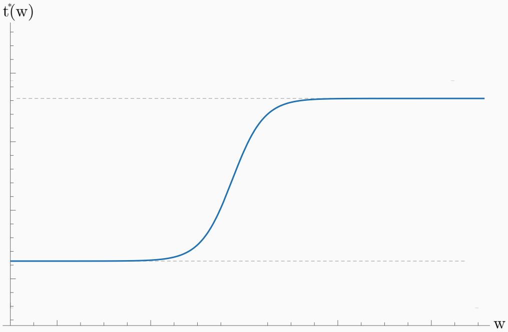

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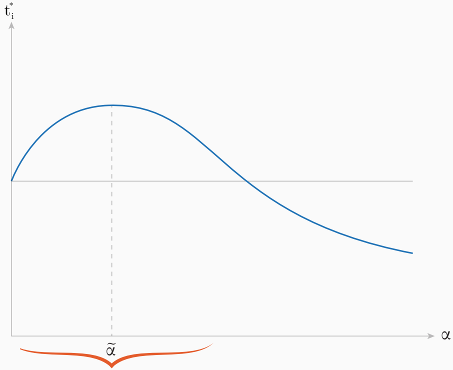

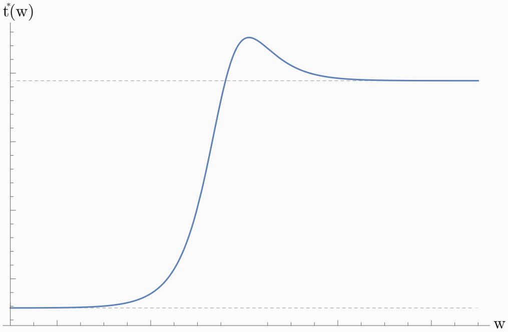

Dynamic Problem

There is an underlying state $\omega \in \{A, B\}$. Unobserved by the agent; agent has a prior.
Agent chooses among two alternatives a or b.
Receives a bonus x if choice matches state, no bonus otherwise.
Agent has initial wealth w.

Information arrives continuously as long as an alternative has not been chosen.
Information: Wiener process dS with drift $\tilde{\mu}(-\tilde{\mu})$ and variance ρ^2 if the state is $A(B)$.
Agent pays flow cost c for information collection.

$$u(a, t|\omega = A) = u(w + x - ct) \qquad u(a|\omega = B) = u(w - ct)$$

$$u(\cdot) = CARA_{\alpha}(\cdot)$$

Ito's lemma leads to

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \theta} \mu \frac{e^{\theta} - 1}{e^{\theta} + 1} + \mu \frac{\partial V}{\partial \theta^2}.$$

Value matching (boundary condition)

$$V\left(z^*(t), t\right) = p\left(z^*(t)\right) \frac{1 - e^{-\alpha(w+x-ct)}}{\alpha} + \left(1 - p\left(z^*(t)\right)\right) \frac{1 - e^{-\alpha(w-ct)}}{\alpha}$$

+Smooth pasting condition
+Initial value condition

$$V(\theta, t|z) = \frac{1}{\alpha} - \frac{e^{\alpha ct}\left(\text{sech}\left(\frac{\theta}{2}\right)e^{-\frac{1}{2}\alpha(2w+x)}\,\textbf{cosh}\left(\frac{1}{2}(z-\alpha x)\right)\textbf{cosh}\left(\frac{1}{2}\theta\sqrt{1-\frac{4\alpha c}{\mu}}\right)\text{sech}\left(\frac{1}{2}z\sqrt{1-\frac{4\alpha c}{\mu}}\right)\right)}{\alpha}$$

Let $z^*(\alpha)$ be the value that sets $\frac{\partial V(z|\theta)}{\partial z} = 0$.

$$\textbf{sinh}\left(\frac{1}{2}\left(z^* - \alpha x\right)\right) - \sqrt{1-\frac{4\alpha c}{\mu}}\,\textbf{cosh}\left(\frac{1}{2}\left(z^* - \alpha x\right)\right)\textbf{tanh}\left(\frac{1}{2}z^*\sqrt{1-\frac{4\alpha c}{\mu}}\right) = 0$$

z^* is a function of only α, x, c and μ.

From the implicit function theorem

$$\frac{\partial z^*}{\partial \alpha} > 0 \quad if\ \alpha < \tilde{\alpha} \qquad\qquad \frac{\partial z^*}{\partial \alpha} \leq 0 \quad if\ \alpha \geq \tilde{\alpha}$$

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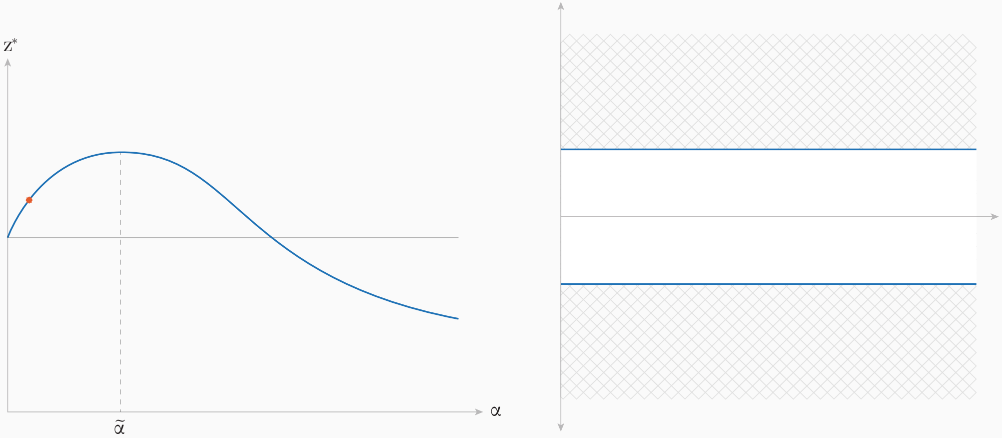

$$u(\cdot) = \sum_i \gamma_i CARA_{\alpha_i}(\cdot) \qquad \gamma_i \geq 0 \qquad \sum_i \gamma_i = 1$$

Without loss assume $\alpha_i \leq \alpha_{i+1}$. Let $V_i(z,t)$ be the value function for α_i

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial \theta}\mu\frac{e^{\theta}-1}{e^{\theta}+1} + \mu\frac{\partial V}{\partial \theta^2}.$$

Value matching (boundary condition)

$$V\left(z^*(t),t\right) = p\left(z^*(t)\right)\sum_{i=1}^{n}\gamma_i\frac{1-e^{-\alpha_i(w+x-ct)}}{\alpha} + \left(1-p\left(z^*(t)\right)\right)\sum_{i=1}^{n}\gamma_i\frac{1-e^{-\alpha_i(w-ct)}}{\alpha}$$

+Smooth pasting condition
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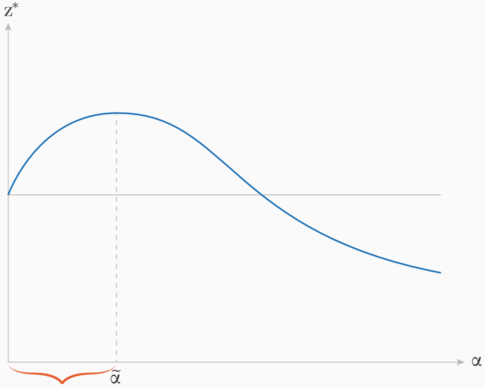
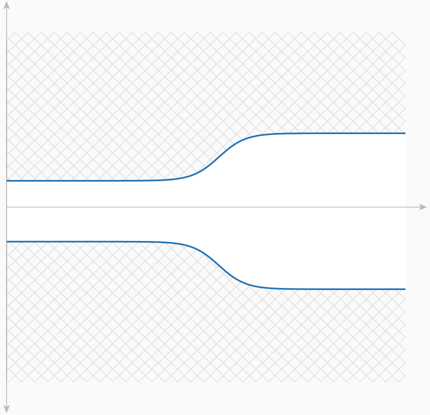

z^*
α
$\tilde{\alpha}$



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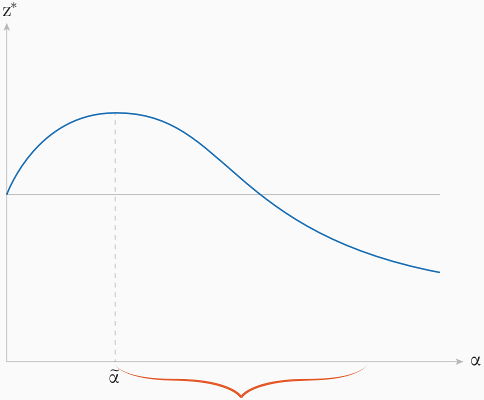

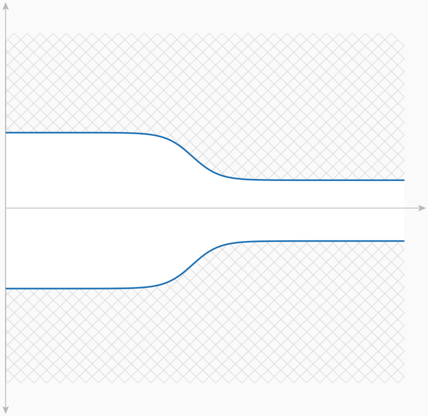

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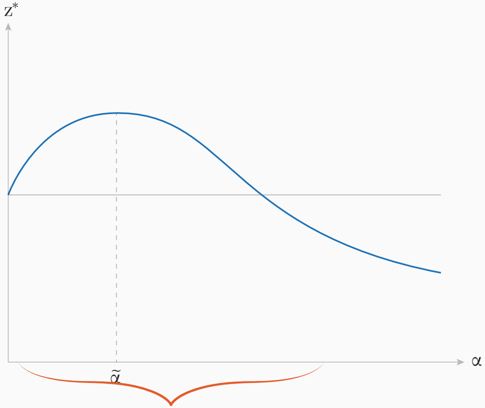

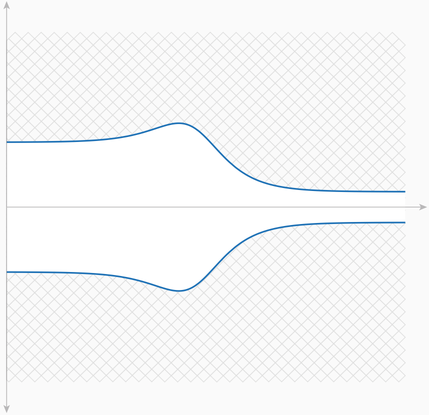

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Possible Desirable Features of this Setup

Paying a fixed price for information of fixed precision Example, buying a 3$ news paper.

What if we pull ct out of $u(\cdot)$: curvature for w and x, but pay for information in utiles.

Then, let $\tilde{u} = u(w)$, re-normalize payments $\tilde{x} = u(w + x) - u(w)$.

Win: $\tilde{u} + \tilde{x}$ Lose: $\tilde{u}$

The problem reduces to the linear problem $\implies$ time-independent boundaries.

From mixed-CARA to any concave $u(\cdot)$

Next Steps

- Approximate continuous, concave and increasing functions with linear combinations of CARAs.

Approximating Concave-Increasing functions

Let $X \subset \mathbb{R}$ be a compact set, and $\mathcal{F}$ be the space of all continuous, increasing and concave functions $f : X \to \mathbb{R}$. Define:

$$C = \left\{ f \in \mathcal{F} : f(x) = -\sum_{i=1}^{N} \beta_i e^{-\alpha_i x}, (\alpha_i, \beta_i) \in \mathbb{R}_+ \times \mathbb{R} \right\}.$$

Apply Stone-Weierstrass

Lemma

C is dense in $\mathcal{F}$ in the uniform norm.

Proof: **An application of Stone-Weierstrass.**

Next Steps

- Approximate continuous, concave and increasing functions with linear combinations of CARAs.✓
- Prove value functions can also be approximated (operator continuity — Shiryaev).

Approximating Value Functions.

Following Shiryaev (2008), define:

$$Tu(\theta,t) = \mathbb{E}_\pi u(\theta_1, t+1) \qquad Qu(\theta,t) = \max\{u(\theta,t), Tu(\theta,t)\} \qquad V_u(\theta,t) = \lim_{n\to\infty} Q^n u(\theta,t).$$

Lemma

For any compact set X and $\varepsilon > 0$, let $||u - w||_X < \varepsilon$, then, $||V_u - V_w||_X < \varepsilon$.

Next Steps

- Approximate continuous, concave and increasing functions with linear combinations of CARAs.✓
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- Show compactness is not a restriction (Shiryaev).

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- Approximate continuous, concave and increasing functions with linear combinations of CARAs.✓
- Prove value functions can also be approximated (operator continuity — Shiryaev). ✓
- Show compactness is not a restriction (Shiryaev).✓
- Derive properties of boundaries for mixes of linear combinations of CARAs.

Summary

- Concave $u \implies$ rich set of boundaries.
- Structure for mixed-CARA.
- Soon: structure for any u.

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